Int. J. Heat Mass Transfer. Vol. 32, No. 12, pp. 2309–2317, 1989
Printed in Great Britain

0017-9310/89 \$3.00+0.00
Pergamon Press plc

A swirled jet problem

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(*Received* 11 *April* 1988)

Abstract—The solution for the problem of a swirled viscous incompressible liquid jet issuing from a point source into the space flooded with the same, but quiescent liquid is examined. Possible problem formulations within the framework of the boundary layer theory and the characteristic behaviour of solutions are studied. Exact and approximate analytical expressions are constructed.

1. INTRODUCTION

PERHAPS no other problem in the theory of viscous jets has attracted the attention of scientists as much as the problem of the development of a submerged swirled axisymmetric jet. Beginning with the fundamental research [1–3], it has been considered both within the framework of the boundary layer theory [4–6] and based on the Navier–Stokes equations [7–10]. However, notwithstanding the fact that there has accumulated a voluminous bibliography on the subject, including special books [11, 12], the mathematical analysis of the problem has not as yet been exhausted because of significant difficulties. The results obtained, highly ambiguous at times, testify to a very complex nature of the flow, which is why it does not readily lend itself to theoretical analysis.

Taking into account the fact that interest remains strong in the problem, this paper presents the results of a theoretical investigation into the development of a submerged swirled jet (with and without consideration for buoyancy forces) carried out using the system of boundary layer equations under the Boussinesq approximation. A great deal of attention is given to the analysis of the behaviour of different analytical solutions and to the range of applications of the theory, since it is just the formulation of the problem and approaches to its solution which have many points open to question and which give rise to contradictory views.

2. BASIC EQUATIONS

The problem is posed for the following system of the boundary layer equations:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial \Delta p}{\partial x}+\frac{\nu}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)+g\beta\Delta T$$

$$\frac{\partial}{\partial x}(yu)+\frac{\partial}{\partial y}(yv)=0,\quad \frac{w^2}{y}=\frac{1}{\rho}\frac{\partial \Delta p}{\partial y}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{vw}{y}=\frac{\nu}{y^2}\frac{\partial}{\partial y}\left[y^2\left(\frac{\partial w}{\partial y}-\frac{w}{y}\right)\right]$$

$$u\frac{\partial \Delta T}{\partial x}+v\frac{\partial \Delta T}{\partial y}=\frac{a}{y}\frac{\partial}{\partial y}\left(y\frac{\partial \Delta T}{\partial y}\right) \quad (1)$$

with the boundary conditions

$$y=0:\quad v=\frac{\partial u}{\partial y}=\frac{\partial \Delta T}{\partial y}=w=0$$

$$y\to\infty:\quad u,\Delta T,w,\Delta p\to 0 \quad (2)$$

and integral conditions

$$Q_0=2\pi\rho C_p\int_0^\infty u\Delta T y\,\mathrm{d}y=\text{const.} \quad (3)$$

The theoretical analysis of the above system is difficult because of its obvious nonlinearity and nonlocality. The latter results from the dependence of the velocity vector components u and v at each point on the values w and ΔT. These facts prevent the problem from being solved by conventional methods. In this case the most natural method of solving the problem is based on the use of different approximate computational methods which allow one to find, though not always accurately enough, the laws governing the phenomenon studied. The methods will not be described here in detail, their review can be found elsewhere [13]; it should only be pointed out that, being distinguished by the mathematical rigour and generality, they are very diversified: from integral methods to the method of asymptotic expansions, with the solution of the problem being finally reduced to results which only differ slightly from one another.

3. A SWIRLED SUBMERGED JET

The foundations of the theory of forced swirled jets were laid by Loitsyanskiy [1]. He suggested an approach according to which one of the velocity vector components, azimuthal, was considered to decrease considerably more quickly than the other two, longitudinal and radial. Taking this fact as a basis, it is

† Deceased.

NOMENCLATURE

- a thermal diffusivity coefficient
- C_p specific heat at constant pressure
- g gravitational acceleration
- K momentum of jet
- L moment of momentum of jet
- $\dot{m}$ mass flow rate per second
- p pressure
- Q excessive heat content flux
- T temperature, $\Delta T = T - T_\infty$
- u, v, w axial, radial and tangential velocity vector components, respectively
- x vertical coordinate
- y radial coordinate.

Greek symbols

- β coefficient of volumetric expansion
- ν kinematic viscosity coefficient
- ρ density.

Subscripts

- c value of the axis
- m maximum value
- 0 initial value
- ∞ parameters at infinity, in surrounding medium.

possible to construct the solution for the problem in the form of an asymptotic series

$$\psi \sim \nu\{f_0 x + f_1 + f_2 x^{-1} + \ldots\}. \tag{4}$$

The first term gives the field of velocities for a non-swirled jet and the following terms allow for the swirl effect which increases as the jet source is approached. In general, at first the ideas of his paper had a marked effect on the trend of nearly all theoretical studies by having given a clue to a distinct differentiation of investigations—all of them were aimed primarily at obtaining higher approximations [4]. This, in turn, led to the problem of large perturbations or, in other words, to the problem of close coupling [14]. One of its most essential, and even fundamental, findings is that series (4) presented above cannot satisfy all of the boundary conditions (2). Moreover, the singularities appearing in higher-order terms not only remain, but get even more pronounced. This is responsible, in the authors' opinion, for a noticeable decrease of the interest in this kind of theoretical investigations.

The nonuniformities occurring when solving the problem in the form of asymptotic series (4) physically manifest the interaction between the fields of u and w in a system and ultimately are due to the complexity of the jet boundary layer structure characterized by the presence of the backflow zone. Therefore, there is a widely held view in the literature that the present problem does not have a closed-form solution.

Because of the absence of any considerable advance in the development of analytical approaches to the solution of the problem, focus of the researchers' attention has swung to numerical simulation [7, 15]. However, the calculations revealed the bifurcations of the solution for certain flow regimes. Clearly, this restricts the possibility of the numerical analysis drastically. Therefore, the situation is such that any significant advances can be made only with the use of the interrelated analytical and numerical methods. It is highly desirable that the method be developed for solving a system of coupled non-linear equations and for finding, on its basis, exact closed-form relations. This also provides the possibility for a constructive assessment of various approximate theories—a very urgent and important aspect of the problem, which very often is responsible for the a priori judgement on the adequacy of the mathematical model and of the jet transfer processes and, in the cases of more complex systems, of the mathematical model itself.

An attempt will now be made to construct an exact solution to the problem of a swirled jet. Thus, the solution is sought for equations (1) (at $\beta = 0$) which satisfy boundary (2) and integral

$$L_0 = 2\pi\rho \int_0^\infty u w y^2 \, dy = \text{const.} \tag{5a}$$

conditions in the class of the form

$$\psi = \nu f(\eta) p(x), \quad w = \frac{L_0}{4\pi\mu}\left(\frac{K_0}{\pi\rho\nu^2}\right)^{1/2} b(\eta)\omega(x)$$

$$\frac{\Delta p}{\rho} = \frac{L_0^2 K_0}{16\pi^3\rho^3\nu^4} a(\eta)\kappa(x), \quad \eta = \frac{K_0}{4\pi\rho\nu^2} y^2 q(x). \tag{5b}$$

Applying to system (1) the formulae which transform the old variables (x, y) into the new ones (x, η) and taking into account that

$$u = \frac{K_0}{2\pi\mu} f_\eta p q, \quad v = -\frac{\nu}{y}\left(f p_x + f_\eta \eta \frac{p q_x}{q}\right)$$

obtain

$$(\eta f_{\eta\eta})_\eta p q^2 + \tfrac{1}{2} f f_{\eta\eta} p_x p q^2 - \tfrac{1}{2} f_\eta^2 p q (pq)_x - 2\Omega\left(a\kappa_x + a_\eta \eta \frac{\kappa q_x}{q}\right) = 0$$

$$(\eta b_\eta)_\eta \omega q + \tfrac{1}{2} f b_\eta p_x \omega q - \tfrac{1}{2} f_\eta b p q \omega_x + \tfrac{1}{4} f_\eta b p \omega q_x + \frac{1}{4\eta} f b p_x \omega q - \frac{1}{4\eta} b \omega q = 0$$

$$2 a_\eta \eta \kappa = b^2 \omega^2, \quad \Omega = \frac{L_0^2}{16\pi\rho\nu^2 K_0} \tag{6}$$

under the boundary conditions

$$f(0)=0,\quad \lim_{\eta\to 0}\sqrt{\eta}f_{\eta\eta}=0,\quad f_\eta(\infty)=0$$

$$b(0)=0,\quad b(\infty)=0,\quad a(\infty)=0. \tag{7}$$

Moreover, the same calculations yield the solution non-triviality conditions

$$\left(\int_0^\infty f_\eta b\eta^{1/2}\,d\eta\right)p\omega q^{-1/2}=1. \tag{8}$$

To specify the integration scheme for the system of equations (6)–(8), assume that

$$pq^2=c_1(1+n\varphi),\quad p_x pq^2=c_2(1+n\varphi)$$

$$pq(pq)_x=c_3(1+k\varphi),\quad \varphi=-2\Omega\kappa_x$$

$$d=\left[a+a_\eta\eta\frac{\kappa q_x}{\kappa_x q}\right]. \tag{9}$$

Then, equation $(6)_1$ will transform into

$$c_1(\eta f_{\eta\eta})_\eta+\tfrac{1}{2}c_2 ff_{\eta\eta}-\tfrac{1}{2}c_3 f_\eta^2=0 \tag{10}$$

$$c_1 n(\eta f_{\eta\eta})_\eta+\tfrac{1}{2}c_2 nff_{\eta\eta}-\tfrac{1}{2}c_3 kf_\eta^2+d=0. \tag{11}$$

The solution of equation (10) has the form

$$f=\frac{4\alpha\eta}{1+\alpha\eta},\quad f_\eta=\frac{4\alpha}{(1+\alpha\eta)^2}. \tag{12}$$

It was found while integrating that $c_2=-c_3$ and also assumed, without loss of generality, that $c_1/c_2=1$. This will give $p_x=1$. To the values of coefficients obtained add the expression $p\omega q^{-1/2}=1$ which follows from integral condition (8). Now, the transformation of equations $(6)_2$ and $(6)_3$ will yield

$$(\eta b_\eta)_\eta+\tfrac{1}{2}(fb)_\eta-\frac{1}{4\eta}(1-f)b=0$$

$$2a_\eta\eta=b^2. \tag{13}$$

In view of relation (8), the solution of equation (13) can be written in the form

$$b=\tfrac{2}{3}\sqrt{\alpha}\frac{\sqrt{(\alpha\eta)}}{(1+\alpha\eta)^2},\quad a=-\tfrac{3}{8}\alpha\frac{1}{(1+\alpha\eta)^3}. \tag{14}$$

Returning to equation (11), obtain

$$c_1 n(\eta f_{\eta\eta})_\eta+\tfrac{1}{2}c_2 nff_{\eta\eta}+\tfrac{1}{2}c_2 kf_\eta^2+d=0. \tag{15}$$

Integrating further expression (15) from 0 to ∞ find

$$(-\tfrac{1}{2}c_2 n+\tfrac{1}{2}c_2 k-\tfrac{1}{8})\int_0^\infty f_\eta^2\,d\eta=0$$

which yields $k=n+1/(4c_2)$. Consequently, the following equation is obtained for finding the functions p, q, κ, ω:

$$pq(pq)_x+p_x pq^2=\tfrac{1}{2}\Omega(q/p^2)_x$$

$$p_x=1,\quad \kappa=q/p^2,\quad \omega=q^{1/2}/p. \tag{16}$$

It can be verified that the following relations ensure the necessary coupling of relations (16):

$$q=\frac{1}{(2)^{2/3}}x^{-2}[(1+\sqrt{(1-\tfrac{1}{2}\zeta^3)})^{1/3}+(1-\sqrt{(1-\tfrac{1}{2}\zeta^3)})^{1/3}]^2$$

$$\omega=\frac{1}{(2)^{1/3}}x^{-2}[(1+\sqrt{(1-\tfrac{1}{2}\zeta^3)})^{1/3}+(1-\sqrt{(1-\tfrac{1}{2}\zeta^3)})^{1/3}]$$

$$\kappa=\frac{1}{(2)^{2/3}}x^{-4}[(1+\sqrt{(1-\tfrac{1}{2}\zeta^3)})^{1/3}+(1-\sqrt{(1-\tfrac{1}{2}\zeta^3)})^{1/3}]^2$$

$$p=x,\quad \zeta=\Omega x^{-2}=\frac{L_0^2}{16\pi\rho v^2 K_0}x^{-2}. \tag{17}$$

Thus, for the velocity component u, w, and the quantity $\Delta p/\rho$ of an axisymmetric laminar jet, obtain

$$u=\frac{K_0}{\pi\mu}\frac{2\alpha}{(1+\alpha\eta)^2}pq$$

$$w=\frac{L_0}{4\pi\mu}\left(\frac{K_0}{\pi\rho v^2}\right)^{1/2}\tfrac{3}{2}\sqrt{\alpha}\frac{\sqrt{(\alpha\eta)}}{(1+\alpha\eta)^2}\omega$$

$$\frac{\Delta p}{\rho}=-\frac{L_0^2 K_0}{16\pi^3\rho^3 v^4}\tfrac{3}{8}\alpha\frac{1}{(1+\alpha\eta)^3}\kappa. \tag{18}$$

The integral condition

$$K_0=\lim_{x\to\infty}\left(2\pi\rho\int_0^\infty u^2 y\,dy\right)$$

(i.e. here the quantity K_0 stands for the axial momentum flux at an infinite distance from the jet source) gives

$$\alpha=\tfrac{3}{16}. \tag{19}$$

It should be noted that the quantity d in equation (15) was replaced in the integration by

$$d=-\tfrac{1}{8}f_\eta^2.$$

It is easy to check that this relation is exact at $\eta=0$ and approximate, satisfactorily approximating the exact curve, at $\eta\neq 0$.

Thus, the self-similar solution for the problem of submerged swirled jet development is constructed, which, according to the aforegoing, is exact on the axis, i.e. for the maximum characteristics, and is approximate in the region $\eta\neq 0$. For $x\to\infty$, the formulae are available which were obtained earlier by Loitsyanskiy [1] and somewhat later by Görtler [2] (the solutions for u and v are represented by a non-swirled jet [16]). To put it another way, when $x\to\infty$ the swirled jet flow degenerates gradually into a non-swirled one as a consequence of the swirling velocity component decaying more quickly than the rest

$$u\sim x^{-1},\quad v\sim x^{-1},\quad w_m\sim x^{-2},\quad \Delta p/\rho\sim x^{-4}.$$

On the other hand, expressions (18) are valid only within the range

$$0 \leqslant \zeta \leqslant (2)^{1/3}.$$

When $\zeta > (2)^{1/2}$, there are no solutions of the class of functions (5b). Physically, this is quite natural if account is taken of the fact that the region of backflow appears in the vicinity of the jet axis at a certain intensity of swirling (in our case, at a certain value of the parameter ζ). This results in the appearance of 'dips' in the profiles of velocity u. Since formulae (18) do not give this pattern of fluid motion even at $\lambda = (2)^{1/3}$, it can be stated that only those solutions are valid within the framework of the boundary layer theory which satisfy the condition of the existence of the maximum velocity u on the jet axis, that is, which satisfy the relation $u_m = u_c$. In this respect the results obtained distinctly demonstrate the specific properties of the boundary layer. Note that application of formulae (18) to the experimental data of ref. [17] (dealing, naturally, with a turbulent jet) will yield $\zeta = 0.9$–1.15 which virtually does not differ from the theory, $\zeta = (2)^{1/3}$. Furthermore, it is seen from relations (18) that the increase in the rate of swirling is also accompanied by an increase in the axial velocity and in the values of w_m, $\Delta p_m/\rho$. As the parameter

$$\zeta = L_0^2/16\pi\rho\nu^2 K_0 x^2$$

increases, there occurs a decrease in the angle of jet divergence and an increase in the long range of the given jet flow

$$b_u = \text{const.}\frac{x}{[(1+\sqrt{(1-\frac{1}{2}\zeta^3)})^{1/3}+(1-\sqrt{(1-\frac{1}{2}\zeta^3)})^{1/3}]}.$$

Equations (18) are of interest because, on the one hand, they make it possible to track the basic trends on the behaviour of solution on variation of ζ and, on the other hand, they have a direct bearing on the problem of turbulent swirled jet calculation. The thing is that, even though new concepts of turbulence simulation are now being investigated, the present-day practical computations use a traditional approach based on one or other of the closure models for the Reynolds mean boundary layer equations, with the gradient models being widely used to describe free jet flows in the framework of semi-empirical theories. In this case the analysis of a turbulent jet does not virtually differ from the analysis of a laminar flow: the molecular viscosity ν is replaced by the turbulent viscosity ν_t introduced according to one of the Prandtl hypotheses. Being a successful compromise between simplicity and accuracy, this approach formed a constructive foundation for engineering calculations of many simple jet flows [18]. The latter fact and also a purely theoretical advantage of the concept, favoured the now widely held view that it can be used for calculating complex flows and, particularly, turbulent swirled jets. However, having been carried over from laminar to turbulent jets, formulae (18) will not predict even general physical trends in the development of swirled viscous jets. In other words, analysis of the solutions obtained has prompted the conclusion that this mathematical model is entirely inadequate for turbulent swirled flows in terms of one-point moments. Note that some numerical results confirming this standpoint have already been reported [19, 20].

The above solutions can also be used for the analysis of a heat problem. The sought-after temperature field in a swirled jet is determined by the set of equalities

$$\Delta T = \frac{(2Pr+1)Q_0}{8\pi\mu C_p}\frac{1}{(1+\alpha\eta)^{2Pr}}x^{-1}. \tag{20}$$

First of all, here attention is turned to the fact that formula (20) corresponds to the solution for a non-swirled jet [21] (the difference is in the variable η which is equal to $K_0 y^2/4\pi\rho\nu^2 x^2$ at $\zeta = 0$). This means that the effect of swirling (parameter ζ) on the temperature field, which plays the part of a passive impurity, is manifested as the deformation of the ΔT profile. It is different with buoyant jets where this tendency is preserved only at a Prandtl number of 2 [22].

All the above analysis refers to the swirled jet problem which has been solved on the basis of relations (5). The second integral condition, which expresses the conservation law for the moment of jet momentum about the symmetry axis [1], has been derived from the more general expression

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty uwy^2\,\mathrm{d}y = \left[\nu\left(y^2\frac{\partial w}{\partial y}-yw\right)-y^2\overline{vw}\right]_\infty \tag{21}$$

on the assumption about a sufficiently rapid decrease of the velocity component w for $y \to \infty$ ($w \sim 1/y^\varepsilon$, $\varepsilon > 1$). Now, the analysis of the problem will be slightly modified, namely equation (5) will be altered. Instead of the condition $L_0 = \text{const.}$, the sought-after functions should satisfy relation (21). This makes it necessary to solve an eigenvalue problem, since ordinary dimensional considerations are not suitable. Let $\zeta \ll 1$ (the case of weak coupling). This makes it possible to neglect (in the first approximation) the term $\partial\Delta p/\partial x$ in the equation of system (1) and to construct an asymptotic solution. For this, it is assumed that

$$w = Db_i(\eta)x^\lambda, \quad \eta = \frac{K_0}{4\pi\rho\nu^2}\frac{y^2}{x^2} \tag{22}$$

where the function $b_i(\eta)$ satisfies the following equation:

$$(\eta b_i')' + \tfrac{1}{2}f_0 b_i' - \tfrac{1}{2}(\lambda+1)f_0' b_i - \frac{1}{4\eta}b_i(1-f_0) = 0 \tag{23}$$

with boundary conditions

$$b_i(0) = 0, \quad b_i \to 0 \quad \text{for} \quad \eta \to \infty. \tag{24}$$

The parameter λ is unknown beforehand and should be determined. In order to solve the system of equations (23) and (24), it is expedient to go over to new variables of the form

$$z = \frac{\alpha\eta}{1+\alpha\eta}, \quad y_i = \frac{b_i}{\sqrt{(z(1-z))}},$$

$$f_0 = \frac{\alpha\eta}{1+\alpha\eta}, \quad \alpha = \tfrac{3}{16}. \tag{25}$$

In these variables, equation (23) is reduced to a hypergeometric equation

$$z(1-z)y_i''+2(1-z)y_i'-2(\lambda+1)y_i = 0$$

for which the function

$$y_i = c_i F(\gamma,\delta,2,z), \quad \gamma+\delta = 1, \quad \gamma\delta = 2(\lambda+1) \tag{26}$$

is known. It solves the problem at hand in quadratures

$$b_i = c_i \frac{\sqrt{(\alpha\eta)}}{1+\alpha\eta} F\left(\gamma,\delta,2,\frac{\alpha\eta}{1+\alpha\eta}\right) \tag{27}$$

$$\lambda = -\frac{i^2+i+2}{2}, \quad i = 0,1,2,\ldots \tag{28}$$

Note that equations (27) and (28) already involve the condition of the solution nontriviality

$$(\lambda+2)\int_0^\infty f_0' b_i \eta^{1/2}\,d\eta = [2\eta^{3/2}(b_i'-f_0'b_i) - \eta^{1/2}b_i(1-f_0)]_\infty. \tag{29}$$

The spectrum of eigenvalues (28) is of special interest

$$b_i = D\left\{c_0\frac{\sqrt{(\alpha\eta)}}{1+\alpha\eta}x^{-1},\quad c_1\frac{\sqrt{(\alpha\eta)}}{(1+\alpha\eta)^2}x^{-2},\quad c_3\frac{\sqrt{(\alpha\eta)}(1-\alpha\eta)}{(1+\alpha\eta)^3}x^{-4},\ldots\right\}. \tag{30}$$

For example, the first point of the spectrum corresponds to the motion in which the velocity field will be determined by the system of equalities

$$u = \frac{K_0}{\pi\mu}\frac{2\alpha}{(1+\alpha\eta)^2}x^{-1}$$

$$w = 2\pi\Gamma_0\left(\frac{K_0}{\pi\rho\nu^2}\right)^{1/2}\frac{\sqrt{\alpha}}{2}\frac{\sqrt{(\alpha\eta)}}{1+\alpha\eta}x^{-1}$$

$$\frac{\Delta p}{\rho} = -4\pi^2\Gamma_0^2\frac{K_0}{\pi\rho\nu^2}\frac{\alpha}{8}\frac{1}{1+\alpha\eta}x^{-2}, \quad \alpha = \tfrac{3}{16}$$

$$\eta = \frac{K_0}{4\pi\rho\nu^2}\frac{y^2}{x^2}$$

$$\zeta \ll 1, \quad \zeta = \frac{4\pi^3\rho\Gamma_0^2}{K_0}. \tag{31}$$

In this case the quantity yw, with $y \to \infty$, is constant and equal to the azimuthal velocity vector component at ∞, that is

$$yw = 2\pi\Gamma_0 \quad \text{when} \quad y \to \infty.$$

The second point corresponds to the solution of the problem for $L_0 = \text{const.}$ (formulae (18) for $x \to \infty$). The third point (and all the rest) correspond to $L_0 = 0$. Relation (31) differs substantially in character from formula (18) giving $b \sim \eta^{-1/2}$ ($w_m \sim x^{-1}$) for $y \to \infty$, whereas expression (18) yields $b \sim \eta^{-3/2}$ ($w_m \sim x^{-2}$). Thus, the solution constructed above for $\zeta \ll 1$ represents the motion induced by a semi-infinite vortical filament [8], whereas equations (18) correspond to the flow which, starting from ref. [1], has been called a swirled jet. Mathematically this means that there exist two types of 'initial data' for the boundary layer equations (1), such that the solution of system (1) belongs again to this class (self-similar relations). The latter are also the solutions (more precisely, limiting asymptotics) for various model problems. The foregoing analysis is very important because there is an opinion in the literature that these two problems are two different aspects of the same problem. The first is interpreted as a strongly swirled jet whereas the theory based on invariants (5) is related to weakly swirled jets. This is not correct since the solution of generalized problems, to which swirled jets appertain, presupposes the adequacy of the results obtained not only in the asymptotics $\zeta \to 0$, but also in jet flow transition from the regime with $\zeta > 1$ (strong coupling) to the state with $\zeta \ll 1$ (weak coupling). The transition is very smooth, the regions with $\zeta > 1$ and $\zeta \ll 1$ continually replace each other in jet flow. These conditions seem to be satisfied only by the solution of equations (18)–(20). It is interesting to find the latter solution on the basis of a fundamentally different approach, namely in the form of asymptotic expansions. The analysis of expressions (18)–(20) indicates that this procedure will correspond to the representation of the sought-after functions in the form of a series

$$\psi(\zeta,\eta) = \nu x\{f_0+\zeta f_1+\zeta^2 f_2+\ldots\}$$

$$w(\zeta,\eta) = \frac{L_0}{4\pi\mu}\left(\frac{K_0}{\pi\rho\nu^2}\right)^{1/2}x^{-2}\{b_0+\zeta b_1+\zeta^2 b_2\ldots\}$$

in the parameters

$$\zeta = \frac{L_0^2}{16\pi\rho\nu^2 K_0}x^{-2}, \quad \eta = \frac{K_0}{4\pi\rho\nu^2}\frac{y^2}{x^2}$$

which tend to zero with jet development. For example, for the axial velocity component we obtain

$$u = \frac{K_0}{2\pi\mu}x^{-1}\{f_0+\zeta(f_0''\eta+f_0')+\ldots\}.$$

4. A SWIRLED CONVECTIVE JET

The solution to the problem of a convective jet, just as the previous problem of forced jet flow, possesses a significant property of self-similarity. For example, when $\zeta \ll 1$ the introduction of the functions

$$u = \left(\frac{g\beta Q_0}{4\pi\mu C_p}\right)^{1/2} f_0'(\eta), \quad \Delta T = \frac{Q_0}{2\pi\mu C_p}h_0(\eta)x^{-1}$$

$$w = \frac{L_0}{4\pi\mu}\left(\frac{g\beta Q_0}{\pi\mu C_p\nu^2}\right)^{1/4} b_0(\eta)x^{-3/2}$$

$$\frac{\Delta p}{\rho} = \frac{L_0^2}{16\pi^2\mu^2}\left(\frac{g\beta Q_0}{\pi\mu C_p v^2}\right)^{1/2} a_0(\eta)x^{-3} \quad (32)$$

allows the change-over from the equations in partial derivatives (1)–(3) to the system of ordinary differential equations

$$(\eta f''_0)' + \tfrac{1}{2}f_0 f''_0 + h_0 = 0$$

$$\frac{1}{Pr}(\eta h'_0)' + \tfrac{1}{2}(f_0 h_0)' = 0$$

$$(\eta b'_0)' + \tfrac{1}{2}(f_0 b_0)' - \frac{1}{4\eta}b_0(1-f_0) = 0$$

$$2a'_0\eta = b_0^2$$

$$\int_0^\infty f'_0 h_0\,\mathrm{d}\eta = 1, \quad \int_0^\infty f'_0 b_0 \eta^{1/2}\,\mathrm{d}\eta = 1$$

$$f_0(0) = 0, \quad \lim_{\eta\to 0}\sqrt{\eta}f''_0 = 0, \quad f'_0(\infty) = 0, \quad a_0(\infty) = 0$$

$$\lim_{\eta\to 0}\sqrt{\eta}h'_0 = 0, \quad h_0(\infty) = 0, \quad b_0(0) = 0, \quad b_0(\infty) = 0 \quad (33)$$

and thus substantially simplifies the investigation. This class of self-similar solutions (32) (without inclusion of swirling) was indicated by Zeldovich [23] and later by other authors [24, 25]. It was Yih who showed that the system of interrelated equations (33) admits an analytical solution only at two Prandtl numbers equal to 1 and 2. The results of his studies were extended in ref. [26] to the case of swirling

$Pr = 1$

$$f_0 = \frac{6\alpha\eta}{1+\alpha\eta}, \quad f'_0 = \frac{6\alpha}{(1+\alpha\eta)^2}, \quad h_0 = \frac{12\alpha^2}{(1+\alpha\eta)^3}$$

$$b_0 = 2\sqrt{\alpha}\frac{\sqrt{(\alpha\eta)}}{(1+\alpha\eta)^3}, \quad a_0 = \tfrac{2}{5}\alpha\frac{1}{(1+\alpha\eta)^5}$$

$$\alpha = 1/(18)^{1/2}$$

$Pr = 2$

$$f_0 = \frac{4\alpha\eta}{1+\alpha\eta}, \quad f'_0 = \frac{4\alpha}{(1+\alpha\eta)^2}, \quad h_0 = \frac{8\alpha^2}{(1+\alpha\eta)^4}$$

$$b_0 = \tfrac{3}{2}\sqrt{\alpha}\frac{\sqrt{(\alpha\eta)}}{(1+\alpha\eta)^2}, \quad a_0 = -\tfrac{3}{8}\alpha\frac{1}{(1+\alpha\eta)^3}$$

$$\alpha = (5/32)^{1/2}$$

$$\zeta \ll 1, \quad \zeta = \frac{L_0^2}{8\pi^2\mu^2 v}\left(\frac{\pi\mu C_p}{g\beta Q_0}\right)^{1/2} x^{-3}. \quad (34)$$

The cases with other Prandtl numbers require numerical integration, though the use of some approximate methods is also possible [27]. Should the integral condition (5) in equations (1)–(3), (5) be substituted by relation (21), the basic system will again admit an analytical solution. Assuming that the field of velocities u and v is prescribed (formulae (32)) for $\zeta \ll 1$, the problem can be formulated in the following way: find the function

$$w = Db_i(\eta)x^\lambda, \quad \eta = \left(\frac{g\beta Q_0}{16\pi\mu C_p v^2}\right)^{1/2}\frac{y^2}{x} \quad (35)$$

which satisfies the equation

$$(\eta b'_i) + \tfrac{1}{2}f_0 b'_i - \tfrac{1}{2}(\lambda+\tfrac{1}{2})f'_0 b_i - \frac{1}{4\eta}b_i(1-f_0) = 0$$

$$b_0(0) = 0, \quad b_i \to 0 \text{ for } \eta\to\infty \quad (36)$$

and the integral condition

$$(\lambda+\tfrac{3}{2})\int_0^\infty f'_0 b_i \eta^{1/2}\,\mathrm{d}\eta = [2\eta^{3/2}(b'_i - \tfrac{1}{4}f'_0 b_i) - \eta^{1/2}(1-f_0)b_i]_\infty. \quad (37)$$

The analysis of equations (36) and (37) for $Pr = 2$ yields

$$b_i = c_i\frac{\sqrt{(\alpha\eta)}}{1+\alpha\eta}F\left(\gamma,\delta,2,\frac{\alpha\eta}{\alpha\eta+1}\right)$$

$$\gamma+\delta = 1, \quad \gamma\delta = 2(\lambda+\tfrac{1}{2}), \quad \alpha = (5/32)^{1/2}. \quad (38)$$

Direct computation gives

$$\lambda = \frac{i^2+i+1}{2}; \quad i = 0,1,2,\ldots \quad (39)$$

To these eigenvalues there corresponds a system of eigenfunctions

$$b_i = D\left\{c_0\frac{\sqrt{(\alpha\eta)}}{1+\alpha\eta}x^{-1/2}, \quad c_1\frac{\sqrt{(\alpha\eta)}}{(1+\alpha\eta)^2}x^{-3/2},\ldots\right\}. \quad (40)$$

It is not difficult to see that the solution corresponding to $\lambda = -1/2$

$$u = \left(\frac{g\beta Q_0}{4\pi\mu C_p}\right)^{1/2}\frac{4\alpha}{(1+\alpha\eta)^2}, \quad \Delta T = \frac{Q_0}{2\pi\mu C_p}\frac{8\alpha^2}{(1+\alpha\eta)^4}x^{-1}$$

$$w = 2\pi\Gamma_0\left(\frac{g\beta Q_0}{\pi\mu C_p v^2}\right)^{1/4}\frac{\sqrt{\alpha}}{2}\frac{\sqrt{(\alpha\eta)}}{1+\alpha\eta}x^{-1/2}, \quad \alpha = (5/32)^{1/2}$$

$$\frac{\Delta p}{\rho} = -4\pi^2\Gamma_0^2\left(\frac{g\beta Q_0}{\pi\mu C_p v^2}\right)^{1/2}\frac{\alpha}{8}\frac{1}{1+\alpha\eta}x^{-1}$$

$$\zeta \ll 1, \quad \zeta = \frac{8\pi^2\Gamma_0^2}{v}\left(\frac{\pi\mu C_p}{g\beta Q_0}\right)^{1/2}x^{-1} \quad (41)$$

describes the case $L \neq$ const. The second eigenvalue $\lambda = -3/2$ belongs to the problem $L_0 =$ const. It was considered and solved in ref. [26] (formulae (32)). When $\lambda \geqslant -7/2$, the quantity $L_0 =$ const. $= 0$.

The present problem with a substantial interpolation between the fields of u, w, and ΔT is much more complicated than the problem of a forced swirled jet where the velocity and temperature fields could be calculated separately. In other words, in this jet flow the Archimedes forces already act jointly with the rotation effects, and the boundary layer structure depends greatly on their relationship. Since it has been assumed in the analysis that $\zeta \ll 1$, the solutions obtained can be valid only for a situation when the thermogravitation effects on hydrodynamics and heat transfer is at least an order of magnitude higher than the effect of rotation. Therefore, for the case of strong coupling formulae (32) will only yield a qualitative flow pattern valid for $x \to \infty$. To obtain qualitative information, it is necessary to use again the system of equations (1)–(3) and (5).

By introducing, in the usual fashion, the stream function ψ, temperature ΔT, the swirling rate w and the excess pressure $\Delta p/\rho$, it is possible to make an attempt to find the solution of equations (1)–(3) in the form of the asymptotic series

$$\psi(\zeta,\eta) = vx\{f_0+\zeta f_1+\ldots\}$$

$$\Delta T(\zeta,\eta) = \frac{Q_0}{2\pi\mu C_p}x^{-1}\{h_0+\zeta h_1+\ldots\}$$

$$w(\zeta,\eta) = \frac{L_0}{4\pi\mu}\left(\frac{g\beta Q_0}{\pi\mu C_p v^2}\right)^{1/4}x^{-3/2}\{b_0+\zeta b_1+\ldots\}$$

$$\frac{\Delta p}{\rho}(\zeta,\eta) = \frac{L_0^2}{16\pi^2\mu^2}\left(\frac{g\beta Q_0}{\pi\mu C_p v^2}\right)^{1/2}x^{-3}\{a_0+\zeta a_1+\ldots\}$$

$$\eta = \left(\frac{L_0^2}{16\pi\mu C_p v^2}\right)^{1/2}\frac{y^2}{x} \tag{42}$$

expanded in powers of the parameter

$$\zeta\frac{L_0^2}{8\pi^2\mu^2 v}\left(\frac{\pi\mu C_p}{8\beta Q_0}\right)^{1/2}x^{-3}.$$

Substituting expressions (42) into equations (1)–(3) and then grouping the terms of the same order in ζ gives the following system of equations:

zero approximation

$$(\eta f_0'')'+\tfrac{1}{2}f_0 f_0''+h_0 = 0$$

$$\frac{1}{Pr}(\eta h_0')'+\tfrac{1}{2}(f_0 h_0)' = 0$$

$$(\eta b_0')'+\tfrac{1}{2}(f_0 b_0)'-\frac{1}{4\eta}b_0(1-f_0) = 0$$

$$2a_0'\eta = b_0^2; \tag{43}$$

first approximation

$$(\eta f_1'')'+\tfrac{1}{2}f_0 f_1''+\tfrac{3}{2}f_0' f_1'-f_0'' f_1+h_1+3a_0+a_0'\eta = 0$$

$$\frac{1}{Pr}(\eta h_1')'+\tfrac{1}{2}f_0 h_1'+2f_0' h_1 = -\tfrac{1}{2}f_1' h_0+f_1 h_0'$$

$$(\eta b_1')'+\tfrac{1}{2}f_0 b_1'+2f_0' b_1-\frac{1}{4\eta}b_1(1-f_0)$$

$$= -\tfrac{1}{2}f_1' b_0+f_1 b_0'+\frac{1}{2\eta}f_1 b_0$$

$$\eta a_1' = b_0 b_1$$

$$\int_0^\infty (f_0' h_1+f_1' h_0)\,\mathrm{d}\eta = 0$$

$$\int_0^\infty (f_0' b_1+f_1' b_0)\eta^{1/2}\,\mathrm{d}\eta = 0 \tag{44}$$

and so on. This approach, which employs the formulae for a non-swirled convective jet (zero approximation) as the basic solution and which is based on the perturbation method procedure, makes it possible to predict the deviations introduced into an ordinary jet convective flow by swirling as well as their effect on the basic hydrodynamic and thermal characteristics. For this, it is necessary to construct the solution of the problem in the first approximation. Based on the results of ref. [26] at $Pr = 2$, obtain

$$f_1 = \gamma f_0'\eta,\quad f_1' = \gamma(f_0''\eta+f_0')$$

$$h_1 = \gamma h_0'\eta,\quad b_1 = \gamma(b_0'\eta+\tfrac{1}{2}b_0)$$

$$a_1 = \gamma(a_0'\eta+a_0),\ldots \tag{45}$$

where

$$\gamma = \frac{9\sqrt{(10)}}{80}.$$

For other Pr, the integration of the system of equations (44) cannot be reduced to a simple closed form, therefore numerical methods are required for seeking the values of the functions f_1, h_1, b_1, a_1 which determine the sought-after quantities in equations (42).

It is seen from the formulae

$$f_\eta(\zeta,0) = 1.5811+0.5625\zeta+\ldots$$

$$h(\zeta,0) = 1.2500$$

$$b(\zeta,\eta_m) = 0.3063+0.0545\zeta+\ldots$$

$$a(\eta,0) = -0.1482-0.0527\zeta+\ldots$$

$$u = \left(\frac{g\beta Q_0}{4\pi\mu C_p}\right)^{1/2}f_\eta(\zeta,\eta)$$

$$\Delta T = \frac{Q_0}{2\pi\mu C_p}h(\zeta,\eta)x^{-1}$$

$$w = \frac{L_0}{4\pi\mu}\left(\frac{g\beta Q_0}{\pi\mu C_p^2}\right)^{1/4}x^{-3/2}b(\zeta,\eta)$$

$$\frac{\Delta p}{\rho} = \frac{L_0^2}{16\pi^2\mu^2}\left(\frac{g\beta Q_0}{\pi\mu C_p v^2}\right)^{1/2}a(\zeta,\eta)x^{-3} \tag{46}$$

that the growth of the swirling parameter entails a monotonous increase in nearly all of the maximum characteristics except for the function $h(\zeta,0)$ which is less sensitive to the change of ζ. This evidence testifies to the fact that the self-similar solution for a convective swirled jet has the same properties as do relations (18) and (20). It is also obvious that there exists the critical number $\zeta = \zeta^*$ which separates the zones $u_m > u_c$ and $u_m = u_c$ appearing in swirled flow. On the one hand, this fact brings closer the forced and swirled convective jets. On the other hand, account should be taken of the fact that the profiles of the velocities u, v, w result from the development of the temperature profile. Therefore, the deformation pattern of the velocity and temperature structures in a convective boundary layer will depend substantially on Pr. In this respect, the prediction of the behaviour of a jet flow in the field of buoyancy forces tends to be complicated.

One more significant circumstance should be noted. The thing is that in several publications (the survey of which can be found in ref. [13]) attempts were made to use the boundary layer equations for obtaining qualitative and quantitative information about the flow in the region $u_m > u_c$. However, it has been found in the present work that the solution to the system of equations (1)–(3) exists only in the region $u_m = u_c$, and according to equations (18) and (46), the value of the swirling parameter ζ takes on all the values from 0 up to ζ^* (for a forced jet $\zeta^* = (2)^{1/3}$).

The latter indicates that, apart from the actual solution of the problem, it is necessary to study the adequacy and limitations of a particular mathematical model of complex jet flows.

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UN PROBLEME DE JET TOURBILLONNAIRE

Résumé—On examine la solution du problème d'un jet tourbillonnaire de liquide visqueux incompressible qui, d'un point-source, va dans un milieu empli du même liquide mais au repos. On étudie les formulations possibles du problème dans le cadre de la théorie des conditions aux limites et le comportement caractéristique des solutions. On construit des expressions analytiques exactes et approchées.

EIN WIRBELSTRAHLPROBLEM

Zusammenfassung—Für das Problem eines Wirbelstrahls einer zähen inkompressiblen Flüssigkeit, die von einer punktförmigen Quelle aus in einen Raum mit gleicher, aber ruhender Flüssigkeit strömt, wird eine Lösung angegeben. Mögliche Lösungsansätze im Rahmen der Grenzschichttheorie und das charakteristische Verhalten von Lösungen wird untersucht. Es werden sowohl exakte als auch Näherungslösungen entwickelt.

ЗАДАЧА О ЗАКРУЧЕННОЙ СТРУЕ

Аннотация—Рассматривается решение задачи о закрученной струе вязкой несжимаемой жидкости, вытекающей из точечного источника в пространство, затопленное той же, но покоящейся жидкостью. Изучаются возможные постановки задачи в рамках теории пограничного слоя и характерное поведение решений. Построены точные и приближенные аналитические выражения.